Dynamics of Balanced Parentheses, I.
Dyck Squares

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Abstract. The article deals with a strict lexicographic order on the set of balanced parentheses. Elements of the set are ordered first the length of the code, and then in each sign range the sorting is performed based on the ordering in the alphabet of the Dyck words. A strict order generates procedures for identification, indexing and reconstruction of balanced parentheses. Analysis of the dynamics of the Dyck words helped solve the problem of representing the Catalan number as a sum of squares of natural numbers. In this case, the Dyck triangle is considered in different coordinates. In the calculations, we use the Catalan convolution matrix.
A small software service is offered online to calculate or check some results.

Key Words: ranking of Dyck words, Dyck path, Dyck dynamics, Dyck squares, Catalan convolution matrix.

Based on the version 2015 (in Russian)

This work begins a series articles on the dynamics of Dyck words. At this moment the following papers are planned:

Dynamics of balanced parentheses, I. Dyck squares.
Dynamics of balanced parentheses, II. 4D Dyck triangle and its projections.
Dynamics of balanced parentheses, III. Identification of Dyck words.
Dynamics of balanced parentheses, IV. Dyck polynomials.

Unordered sets are practically unsuitable for analysis. So first we will construct a lexicographic order on the set of balanced parentheses.

1 Lexicographic order

In discrete mathematics, the balanced parentheses, or *Dyck words*, are sufficiently known and play an important role [1]. A Dyck word is a system of interrelated elements, a balanced string of left (open) and right (closed) parentheses using the *Dyck language* with just one kind of brackets. The system of related parentheses determines the dynamics of Dyck words, or *Dyck dynamics*.

For the Dyck word, first, brackets are balanced, i.e. the number of left and right parentheses is the same (the *first rule* of Dyck dynamics). And secondly, in the initial subword, the number of right brackets never exceeds the number of left ones (the *second rule*). For any Dyck word of semilength n (there are n left parentheses and n right ones), the second rule of dynamics is equivalent to the known condition for the position r_i of the ith right parenthesis [2]:

$$2i \le r_i \le n + i,\ 1 \le i \le n.$$

A group of consecutive Dyck words is also a Dyck word. The number of Dyck words of semilength n is equal to the nth Catalan number (see OEIS A000108). There are no restrictions to the length of the bracket sequences, so we can talk about the infinity of the set of the Dyck words.

In infinite non-numeric sets, the *lexicographic order* (*ranking* of elements) is always positive. In the case of strict order, a unique index (natural number) is assigned to each element, that is, a one-to-one correspondence between natural numbers and a given sequence is established. This order inevitably generates procedures for indexing and restoring the items of the ranked set. The first procedure computes the index of any element. The inverse procedure allows us to restore the element for a given index.

Additional procedures are possible for navigating the lexicographical series. The navigation procedures are rigidly related to the dynamics of Dyck words. In our opinion, dynamics generates certain navigation procedures, while the latter specify the dynamics.

Formally, we can say that in the natural series, integers are ranked in length of the code. It is also logical to sort Dyck words by the length of codes. As a result, all Dyck words are distributed over *sign ranges*. In such a ranked sequence, or *Dyck row*, elements consisting of *n* pairs of differently directed parentheses fall into the *n*-range, and the number of these elements is equal to the *n*th Catalan number, $C_n = \binom{2n}{n}/(n+1)$. In the 1-range there is only one element, $C_1 = 1$, so the Dyck row begins with the shortest element “()”. In the next 2-range there are two Dyck words “(())” and “()()”, $C_2 = 2$, so we need to choose some order on this and in the following sign ranges. To do this, first of all we need to sort the alphabet symbols.

In the alphabet of natural numbers, the following strict order is established:

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 .$$

You can say that 0 has a minimum weight, and 9 has a maximum weight. Integers do not start with 0, except zero itself. For example, in the 2-range, 10 is the minimum, 99 is the maximum.

Obviously, in the alphabet of Dyck words, only two variants are possible: either the left parenthesis has a minimal weight, or the right one. In the literature, you can find both options. But it is logical to assign a smaller weight to the left parenthesis, since it starts arbitrary Dyck word. So we will sort Dyck words in the ranges, assuming “(” < “)”. There are examples of the corresponding coding of Dyck words [2, 3]. Let's write down the first 4 ranges of Dyck row.

1-range: ().
2-range: (()), ()().
3-range: ((())), (()()), (())(), ()(()), ()()().
4-range: (((()))), ((()())), ((())()), ((()))(), (()(())), (()()()), (()())(), (())(()), (())()(),
()((())), ()(()()), ()(())(), ()()(()), ()()()().

The reader can use the online software service, which allows you to calculate or check some results. This program displays the requested range of Dyck row (up to 500 items).

2 Dyck path

A Dyck word is often connected with a lattice path of *upsteps* of (1, 1), ascending diagonal vectors, and *downsteps* of (1, –1), descending diagonal vectors. The upstep corresponds to the left parenthesis, and the downstep corresponds to the right one. A *Dyck path* stays above the ground level (original line) that joins its initial and terminal points (vertices). Dyck paths are drawn in the first quadrant, each path starts at the origin. In such a lattice, the positions of the parentheses are plotted along the ground level, *i-axis*, and the *unbalance* (the excess number of left parentheses over right ones) is along the *j-axis*. The unbalance j in each ith position can vary from 0 (the first i brackets form a Dyck word) to i (the first i characters are left parentheses).

Example 1. The figure 1 shows the path of Dyck, which is taken from [3]. Under the i-axis, you see the appropriate Dyck word ((()(()()))) of semilength 6 (the 6-range). As you move along the

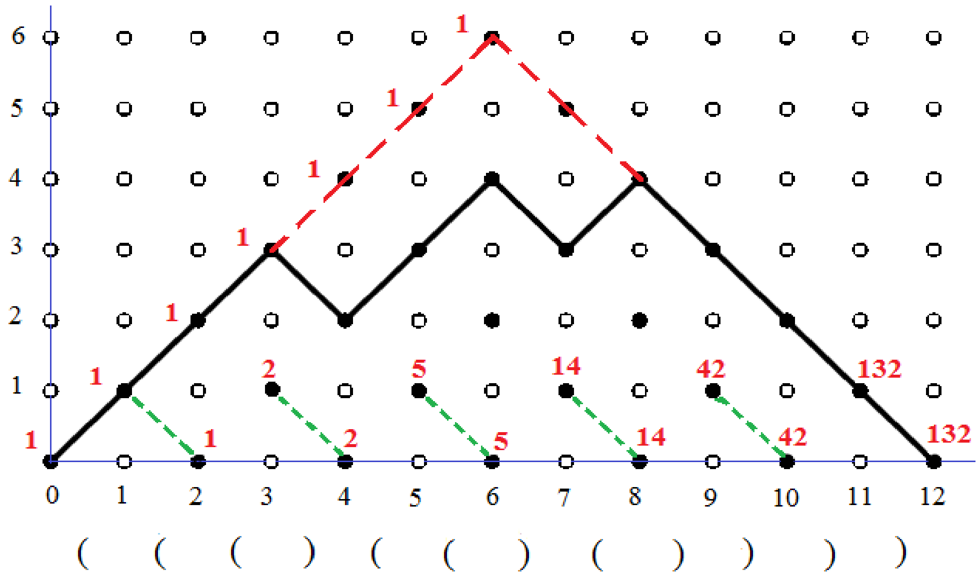

Figure 1.

i-axis, each left parenthesis increases the unbalance (height of the polyline), while the right parenthesis reduces the unbalance. The polyline reaches a height 4. Obviously, the maximum 6 is available only to the first Dyck word (((((()))))). The grid nodes marked with transparent dots are inaccessible (forbidden) for Dyck paths of semilength 6. For any achievable node (i, j), the sum $i+j$ is an even number. ■

Let us call an *n-triangle* figure with boundary points (0, 0), (n, 0) and (2n, 0). A Dyck path of semilength n does not go beyond the n-triangle. The number of such paths is equal to the nth Catalan number. Just so many Dyck paths can be drawn between (0, 0) and (2n, 0). In this connection, the point (12, 0) is marked 132 = C_6 in Figure 1. Similarly, on the i-axis, five nodes are labeled with the previous Catalan numbers 1, 2, 5, 14, 42. Let's calculate the other nodes.

For the node (i, j), label $d(i,j)$ is equal to the number of paths from (0, 0) to (i, j). Obviously, inaccessible nodes are marked by 0 (we omit zeros). It is easy to see that $d(i,i) = 1$, $i \geq 0$, because there is only one path to (i, i) (such paths correspond to adjacent left parentheses in the initial fragment of the Dyck word).

In Figure 1, labels are also shown in the first line ($j = 1$). The group of downsteps (green dashed lines) connects nodes with the same labels, Catalan numbers. To node (2i,0) you can move only from (2i–1, 1), so $d(2i-1,1) = d(2i,0) = C_i$, $i \leq n$.

The considered labels are obvious, let's look at the remaining nodes of the n-triangle. We will analyze the mutual relations of the steps in the paths.

In Figure 2, a group of upsteps connects 1s. Next we have a downstep, and the end node would also get a 1 if it is on the i-axis. But in our case this is not so, and to the end node you can draw an additional upstep from the bottom node labeled, for example, t. As a result, the end node gets $1+t$, because that's how many path we get from both parents.

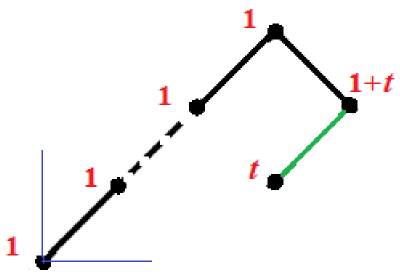

Figure 2.

The described case is typical for each node of the n-triangle. We can lead to a node one upstep and one downstep from neighboring nodes with a smaller i-coordinate, and the label of such node is equal to the sum of labels of the mentioned two nodes. The labels outside the n-triangle are zero, so this rule is valid for all nodes.

Thus, for any node, its label (hereinafter *dynamics*) is determined by the *dynamics equation*:

$$d(i,j) = d(i–1,j–1) + d(i–1,j+1), \quad i \geq j > 0. \tag{1}$$

Using (1) and taking $d(0,0) = 1$, it is easy to calculate the dynamics of all nodes.

3 Dyck triangle and Dyck squares

When Dyck paths are drawn, the abscissas of the diagonal vectors are indicated along the horizontal line. In the ith position of the word Dyck, the imbalance of the brackets does not exceed i, so on the plane, the set of non-zero labels forms the *Dyck triangle* (Figure 3).

8									1		9		54		273
7								1		8		44		208	
6							1		7		35		154		637
5						1		6		27		110		429	
4					1		5		20		75		275		1001
3				1		4		14		48		165		572	
2			1		3		9		28		90		297		1001
1		1		2		5		14		42		132		429	
0	1		1		2		5		14		42		132		429
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Figure 3.

Let's reformat the n-triangle in Figure 1 according to the Dyck triangle. In Figure 4 we showed a new n-triangle, an initial fragment of the Dyck triangle for $n = 6$. It is easy to see that for any node $x = (i, j)$, there is a *symmetric point* $y = (2n–i, j)$ in general case with another dynamics. Further a symmetrical pair or *mirror* x, y, we will denote $x \sim y$.

In the center of the n-triangle, the n-column contains *self-symmetric nodes* (n, j). In Figure 4, the 6th column contains four self-symmetric points 1, 5, 9, 5. It is easy to show that the sum of squares of these numbers is equal to C_6 (indeed, $1^2 + 5^2 + 9^2 + 5^2 = 132$).

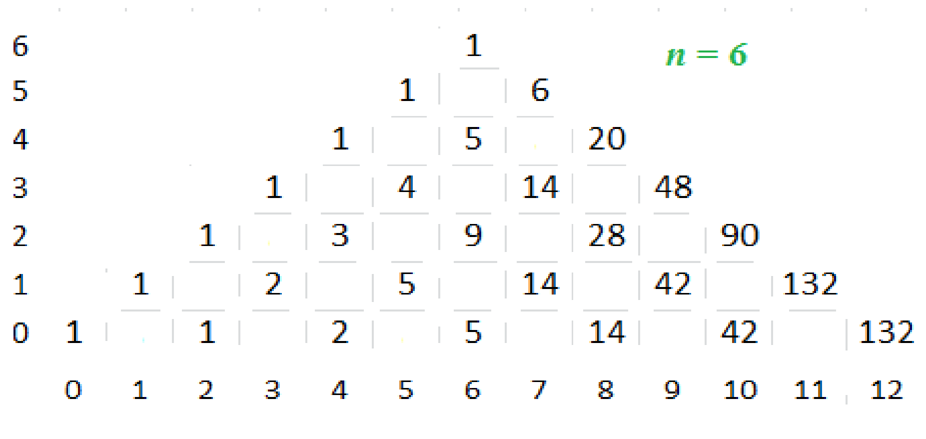

Figure 4.

In the n-triangle, let's invert the Dyck paths by placing their origin at $(2n, 0)$, and recount the *reverse dynamics* $\bar{d}(i, j)$. Then in the mirror $x\sim y$ the nodes are exchanged by the dynamics, that is, $\bar{d}(x) = d(y)$ and $\bar{d}(y) = d(x)$. For self-symmetric node the straight label and reverse label are the same, that is, $d(n, j) = \bar{d}(n, j),\ j \leq n$.

Obviously, the number of Dyck paths through a node x is equal to $d(x)\,\bar{d}(x)$. If we sum of such products for all nodes of any column in the n-triangle, we get the nth Catalan number. This implies the possibility of representing the Catalan number as the sum of squares of integers.

Theorem 1. *The nth Catalan number is equal to the sum of squares of non-zero items of the nth column of the Dyck triangle.*

We will call such squares the *Dyck squares*. It is easy to see that the number of Dyck squares in the sum is $\lfloor n/2 \rfloor + 1$ (the number of elements in the nth column), then

$$C_n = d^2(n, n) + d^2(n, n-2) + \ldots + d^2(n, n-2\lfloor n/2 \rfloor).$$

Let's introduce the notation t_{nk} $(= t_{n,k})$ for the elements of the nth column of the Dyck triangle:

$$C_n = \sum_{k=0}^{\lfloor n/2 \rfloor} t_{n,k}^2, \quad t_{n,k} = d(n, n-2k). \tag{2}$$

In (2) some terms are easy to determine, let's write these:

$t_{n,0} = d(n, n) = 1;$
$t_{n,1} = d(n, n-2) = n - 1;$
$t_{n,2} = d(n, n-4) = 2 + 3 + \ldots + (n-3) + (n-2) = n(n-3)/2;$
$t_{n,\lfloor n/2 \rfloor} = d(n, n-2\lfloor n/2 \rfloor) = C_{\lceil n/2 \rceil}.$

Example 2. We decompose $C_7 = 439$ by the sum of the squares. In this amount, there are $\lfloor 7/2 \rfloor + 1 = 4$ terms. In this case, the terms are defined as follow:

$t_{7,0} = d(7,7) = 1;$ $t_{7,1} = d(7,5) = 7 - 1 = 6;$ $t_{7,2} = d(7,3) = 7(7-3)/2 = 14;$ $t_{7,3} = d(7,1) = C_4 = 14.$

As a result, we get: $C_7 = 1^2 + 6^2 + 14^2 + 14^2 = 439.$ ■

In the next section we will try to obtain a General formula for term t_{nk}.

4 Convolution of Dyck triangle

Let us return to the infinite Dyck triangle (Figure 3). On the horizontal i-axis, the positions of Dyck words are indicated, i.e. an abscissa is the ordinal number of the current sign (left or right parenthesis) of the word. On the vertical j-axis, the unbalance of brackets is indicated, i.e. an ordinate is the excess the number of left parentheses over right ones at the corresponding point on the i-axis. For a Dyck path of semilength n, the i-coordinate takes values from 0 to $2n$, and the j-coordinate takes values from 0 to n. For an arbitrary node (i, j) of the Dyck triangle, the condition $j \le i$ is mandatory and, in addition, $i + j$ = even.

In the Dyck triangle, nodes with the same abscissa are grouped as a column, a vertical line. There are $\lfloor i/2 \rfloor + 1$ elements in the ith column. Nodes with the same ordinate are grouped as an infinite row, a horizontal line. But there is a third way to select similar points; this is to group nodes in a diagonal.

It is easy to see that the nodes (i, j), $i + j = 2n$, are located on the same descending diagonal line (n-*isoline*) with the upper point (n, n) and the lower one $(2n, 0)$. The n-isoline is associated with the nth Catalan number, since the two bottom labels are equal to C_n. In the Figure 5 the 6-isoline is drawn in yellow and “fixed” with the button at the top.

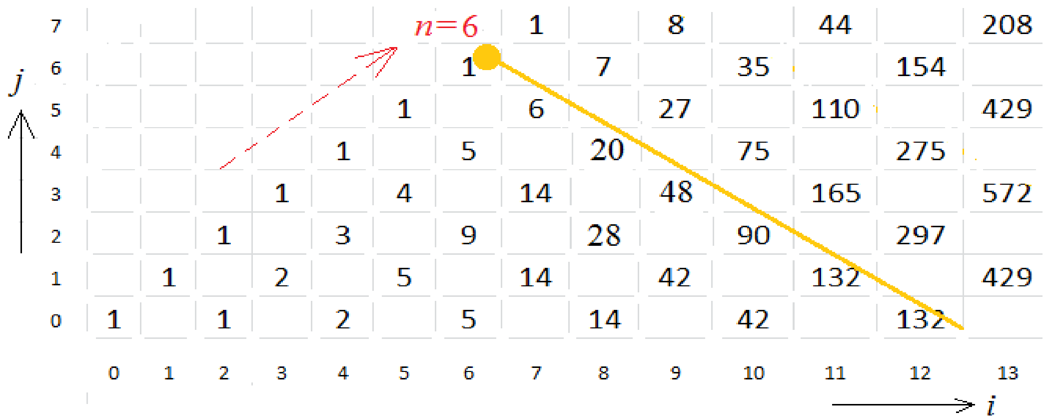

Figure 5.

In the Dyck triangle, each n-isoline is associated with the nth Catalan number. The converse is also true: for each natural number n, we can assign the n-isoline. This means that in addition to the two variables i and j, we can use the third variable n, the index of the Catalan number. In Figure 5, the additional conditional (virtual) axis n is shown with a red dashed line. All three variables are connected by the coordinate equation

$$i + j - 2n = 0, \; 0 \le j \le i \le 2n. \tag{3}$$

In (3), the variable n is, as before, the number of pairs of brackets in Dyck word, but the status of n has grown. Now n is a coordinate along with i and j. The grid $\{i, j\}$ is used in figure 5, so i and j are independent variables, while the variable n is dependent, $n = (i + j)/2$. Now we can say more specifically that in the coordinate grid $\{i, j\}$ we have the Dyck ij-triangle. But the grid can be changed.

Let's transform the Dyck ij-triangle into the Dyck nj-triangle, in other words, we need to change the coordinate grid $\{i, j\}$ to the coordinate grid $\{n, j\}$. This is simply done; it is enough to rotate each k-isoline to the vertical position, fixing the upper node (k, k). The result is a *convolution* of the original Dyck triangle (see Figure 6).

As we see the i-axis is replaced by the n-axis, and in our case the 6-isoline (from Figure 5) is converted in the 6th column. In the new coordinate grid, n and j are independent variables, while the variable i is dependent, $i = 2n - j$. We continue to be interested in the nodes with the same coordinate i, as the labels of these points define Dyck squares in (2). So in Figure 6, the nodes with $i = 6$ are highlighted in green.

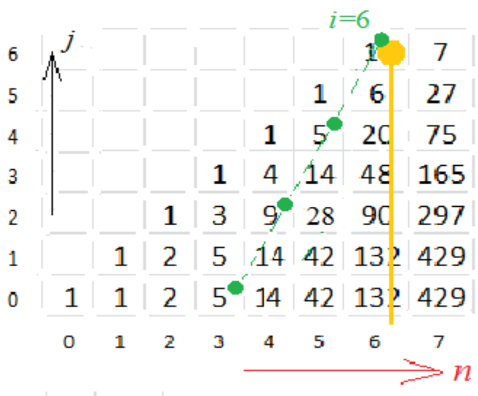

Figure 6.

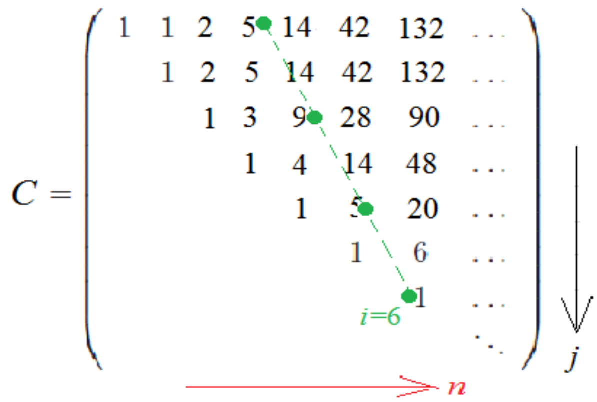

Figure 7.

In the literature, you can find a numerical array similar to Figure 6. Figure 7 shows the known Catalan convolution matrix [4, 5]. The drawing is supplemented with axes n, j, and we highlighted the same four points with the coordinate $i = 6$. In the case $j = 0$ (upper row of the matrix), we get a set of Catalan numbers. It is easy to see that the Dyck nj-triangle and the Catalan convolution matrix are completely identical (the difference is only in the orientation of one axis).

Well-known formulas for Catalan numbers

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}, \quad n \geq 0.$$

Elements of the Catalan convolution matrix (and also of the Dyck nj-triangle) are determined by the following formula [5, p. 2928]:

$$c(n, j) = \binom{2n-j}{n-j} - \binom{2n-j}{n-j-1}, \quad n, j \geq 0. \tag{4}$$

Let's define the relationship between the items of the Dyck ij-triangle and the Dyck nj-triangle. We go through all the highlighted nodes with $i = 6$ (see Figure 6):

$$t_{6,0} = d(6, 6) = c(6, 6) = 1; \quad t_{6,1} = d(6, 4) = c(5, 4) = 5;$$
$$t_{6,2} = d(6, 2) = c(4, 2) = 9; \quad t_{6,3} = d(6, 0) = c(3, 0) = 5.$$

It is not difficult to show that the relationship between the items of arrays is the following:

$$t_{ik} = d(i, j) = c(n, j), \quad n = (i + j)/2, \quad k = (i - j)/2. \tag{5}$$

To the variable i, j, n we added the Dyck square index k. The inverse equalities are interesting

$$i = n + k \text{ and } j = n - k = i - 2k. \tag{6}$$

It is logical to consider (6) as an addition to (3). Figure 8 repeats the Dyck ij-triangle. Here the intersecting diagonals ($n = \text{const}$ and $k = \text{const}$) give a visual representation of the functionality

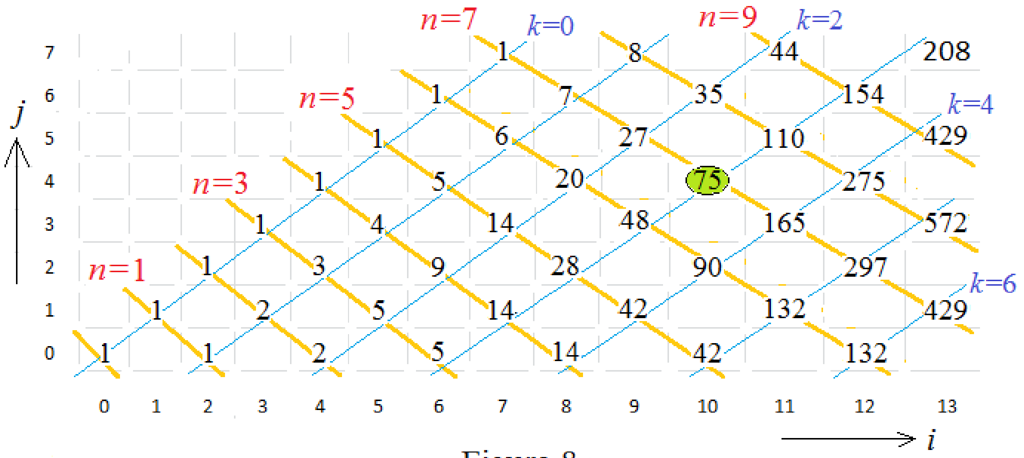

Figure 8.

of the variables n, k. For example, the selected node $x = (10, 4)$ is at the intersection of the diagonals $n = 7$ and $k = 3$. The validity of (3) and (6) is obvious: $i_x = 7 + 3 = 10$, $j_x = 7 - 3 = 4$.

Making use of (4-6) and checking with Figure 6 (the green dashed line), we can write

$$t_{ik} = c(n, j) = c(i-k,\, i-2k) = \binom{2(i-k)-(i-2k)}{(i-k)-(i-2k)} - \binom{2(i-k)-(i-2k)}{(i-k)-(i-2k)-1} = \binom{i}{k} - \binom{i}{k-1}.$$

Thus for the nth Catalan number we get the general formula of the Dyck square terms (2)

$$t_{nk} = \binom{n}{k} - \binom{n}{k-1} = n^{\underline{k-1}}\,(n-2k+1)/k!,\;\; 0 \le k \le \lfloor n/2 \rfloor, \tag{7}$$

where $n^{\underline{k-1}} = n\,(n-1)\cdots(n-(k-2))$ is a falling factorial (k–1 factors).

Note that the equality (7) is true for arbitrary k [6, 7]. For example,

$t_{n,0} = n^{\underline{-1}} \times (n-0+1)/0! = \frac{1}{n+1} \times (n+1)/1 = 1,$

$t_{n,1} = n^{\underline{0}} \times (n-2+1)/1! = 1 \times (n-1)/1 = n-1,$

$t_{n,2} = n^{\underline{1}} \times (n-4+1)/2! = n \times (n-3)/2,$

$t_{n,3} = n^{\underline{2}} \times (n-6+1)/3! = n\,(n-1) \times (n-5)/6,$

$t_{n,4} = n^{\underline{3}} \times (n-8+1)/4! = n\,(n-1)\,(n-2) \times (n-7)/24,$ and so on.

In conclusion, we note that the Dyck ij-triangle and the Dyck nj-triangle (or the Catalan convolution matrix) can be regarded as projections of some spatial construction, 3D Dyck triangle, in three-dimensional grid $\{i, j, n\}$. But we'll talk about this in another article.

5 Online software service

This section describes a small software service that allows any reader to perform simple calculations. The program complex works on the browser side, since only HTML, CSS, JScript components are used.

The first program displays the requested range of Dyck row (up to 500 items) in a table form. In the browser dialog box, the user specifies the range number (the number of pairs of parentheses). Balanced parentheses are displayed line by line. In each line, the starting element is marked with an index in the current range. The number of elements in the rows varies depending on the length of the Dyck words.

The second program indexes the given Dyck word (direct task of identification). First of all, the correctness of the balanced parentheses is checked, and then the program prints the range number and both indexes, relative (in the current range) and absolute (in the Dyck row).

The third program performs the reconstruction of the word Dyck at the specified absolute index (inverse identification problem).

The last fourth program prints a small group of Catalan numbers (up to 10), starting with a given index (not more than 1000).

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