

Factoring a Catalan Number into Chebyshev's Segments

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April 12, 2016

Abstract. This paper describes the “cutting” of the Catalan number decomposition in Chebyshev's segments. The segment content is not calculated and is selected from a set of primes according to the segment bounds. In order to decompose the n th Catalan number it is enough to calculate primes up to $\sqrt{2n}$, i.e. to get a tiny core that corresponds to the factoring a so-called light Catalan number.

Key Words: Prime factorization, factoring, decomposition, Catalan numbers, Chebyshev, Cayley.

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1. Introduction

A Prime Factorization of large numbers takes time to compute. Often factoring special numbers such as Fermat numbers, Mersenne numbers, Catalan numbers, Motzkin numbers, etc. are used to validate methodologies and algorithms for decomposition of huge integers. This paper is related to Catalan numbers that occur in numerous combinatorial applications (see, e.g., [Stan15]).

Cayley formula. Special numbers are characterized by mutual relationships between elements of the corresponding sequences, and this simplifies a Prime Factorization. Relationships are implemented by recurrence and analytical formulas.

Let \mathbb{N} denote non-negative integers. The n th Catalan number is defined so

$$C(n) = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \quad n \in \mathbb{N}. \quad (1)$$

The first Catalan numbers for $n = 0, 1, 2, 3, \dots$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, ... (see [A000108]).

We are interested in a composite Catalan number $C(n) > 5$, i.e. $n > 3$. So then everywhere index $n \geq 4$. In (1) we can reduce a number of operands. Let's separate even factors from odd factors in the numerator

$$(2n)! = 2 \cdot 4 \cdot 6 \cdots 2n \times 1 \cdot 3 \cdot 5 \cdots (2n-1) = 2^n \times n! \times (2n-1)!!$$

After the cuts it will receive the dependency (see [Wei16])

$$C(n) = 2^n \times (2n-1)!! / (n+1)! \quad (2)$$

A similar expression was published by Arthur Cayley in 1859, counting a number of binary trees (see historical review [Pak14]). So let us call the equation (2) *Cayley formula*. The combination of an ordinary factorial and an odd double factorial simplifies a Prime factorization of Catalan numbers.

Let's repeat the procedure with the ordinary factorial

$$(n+1)! = 2 \cdot 4 \cdot 6 \cdots 2 \left\lfloor \frac{n}{2} \right\rfloor \times 1 \cdot 3 \cdot 5 \cdots (2 \left\lfloor \frac{n}{2} \right\rfloor + 1) = 2^{\lfloor n/2 \rfloor} \times \left\lfloor \frac{n}{2} \right\rfloor! \times (2 \left\lfloor \frac{n}{2} \right\rfloor + 1)!!$$

Obviously, $2 \lceil \frac{1}{2} n \rceil$ and $2 \lfloor \frac{1}{2} n \rfloor + 1$ are the largest even integer and the largest odd integer, respectively, that do not exceed $n+1$. Since $\lceil \frac{1}{2} n \rceil + \lfloor \frac{1}{2} n \rfloor = n$, then

$$C(n) = 2^{\lfloor \frac{1}{2} n \rfloor} \times (2 \lfloor \frac{1}{2} n \rfloor + 3) \cdot (2 \lfloor \frac{1}{2} n \rfloor + 5) \cdots (2n-1) / \lceil \frac{1}{2} n \rceil! \quad (3)$$

The expression (3) is convenient and practical. For example, in order to calculate a number of factors of 2 it is sufficient to decompose the factorial $\lceil \frac{1}{2} n \rceil!$

In equations (1–3) all factors are less than $2n$. Let \mathbb{P} denote the set of prime numbers. For the n th Catalan number, let $\text{FB}(n) = {}_p(1, 2n) \subset \mathbb{P}$, the *prime interval*, i. e. the open subset of prime numbers up to $2n$, and let us call this interval the *Factor Base* of $C(n)$. Let F_n denote the multiset of all prime factors of $C(n)$. In general, $F_n \not\subset \text{FB}(n)$ due to prime powers (multiple primes).

Chebyshev's segments. Composition of primes in the multiset F_n is mainly determined by the binomial coefficient $\binom{2n}{n}$. In the middle of the 19th century Chebyshev drew attention to the interval ${}_p(n, 2n) \subset \mathbb{P}$ [Pom15]. This open set of primes is fully included in the Prime Factorization of the central binomial coefficient. Similarly, each prime number from the open interval ${}_p(n+1, 2n) \subset \text{FB}(n)$ is included into F_n . Let's call such intervals *Chebyshev's segments*. State the obvious but important theorem.

Theorem 1.1. *The Prime Factorization of the n th Catalan number contains exactly once each prime from the Chebyshev's segment ${}_p(n+1, 2n) \subset \mathbb{P}$.*

This paper describes the family of noncrossing Chebyshev's segments in the decomposition of the n th Catalan number; the number of these segments less than $\sqrt{n/2}$. The interval ${}_p(n+1, 2n)$ is most extensive, so let's call it the *main segment*. Other Chebyshev's segments are reduced as we move down the base $\text{FB}(n)$.

Gaps between the Chebyshev's segments are closed subsets of \mathbb{P} , a sort of *no-go zone* in which there are no prime factors of the Catalan number. No-go zones are also reduced during the descent down to $\text{FB}(n)$. The longest no-go zone is convenient to consider the infinite set of primes ${}_p[2n, \infty)$. The bounds between Chebyshev's segments and no-go zones are defined using the Catalan number indexes.

We call the family of Chebyshev's segments a *train* of the Catalan number and the vast majority of primes are combined in this train. At the same time, the area of multiple primes or a *core* of the n th Catalan number is compressed to size of the prime interval ${}_p(1, \sqrt{2n})$. Thus almost all base $\text{FB}(n)$ can be divided into Chebyshev's segments.

Example 1.1. To decompose the 1000th Catalan number it is sufficient to calculate only 14 primes from the interval ${}_p(1, \sqrt{2000})$. Other prime factors are selected from Chebyshev's segments. For example, the main segment ${}_p(1001, 2000)$ "furnishes" 135 primes into the multiset F_{1000} . From the adjacent Chebyshev's segment ${}_p(1001/2, 2000/3)$ transferred 26 primes. In the no-go zone ${}_p[2000/3, 1001]$ there is no prime numbers that divide $C(1000)$, i.e. $F_{1000} \cap {}_p[2000/3, 1001] = \emptyset$. In the multiset F_{1000} there are 214 primes and prime powers. The Chebyshev's segments cover 197 primes, and only 17 prime factors fall into the core of $C(1000)$.

Interest is a so-called *light Catalan number*, the prime factorization of which is a tiny core – area of prime powers of the corresponding "heavyweight". At the end of this paper will talk a little bit about light Catalan numbers.

2. Factor base of Catalan Numbers

In general, the multiset F_n has primes and prime powers. All primes from the main segment ${}_p(n+1, 2n) \subset \text{FB}(n)$ are not repeated into F_n . Obviously, the numeric space of $\text{FB}(n)$ can be divided into two disjoint areas. The *upper base* $\text{UB}(n)$ contains primes that cannot theoretically be repeated into F_n . The prime powers and some primes are selected from the *root base* $\text{RB}(n)$. Let $\text{ppb}(n)$ denote the *prime power border*. It could be the bottom of the main segment if the segment was the only one. Below we will show that it is not, and in general, $\text{ppb}(n) \ll n+1$.

Everybody can be found the power border in a Prime Factorization of any natural number. In order to factor a positive integer m sufficient to verify primes up to \sqrt{m} . Only in this range a prime can theoretically be repeated, so in this case the power border is equal to \sqrt{m} .

No-go zones. Consider in the upper base $\text{UB}(n)$ those primes that are adjacent to the bottom of main segment ${}_p(n+1, 2n)$. Let's write in detail the factorials in Cayley formula (2):

$$C(n) = 2^n \times \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)(2n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n(n+1)}. \quad (4)$$

In the denominator the last factor $n+1$ may be a prime in the case of even n . But this factor is in the numerator, and also in a single copy. So $n+1 \notin F_n$. The odd factor n is also available in a single copy in the both factorials, so $n \notin F_n$. The situation persists if we continue to descend on natural numbers until we get to $\frac{2}{3}n$.

This number may be an integer, and therefore even and composite. So, $\frac{2}{3}n \notin F_n$. Thus any prime $\frac{2}{3}n < q \leq n+1$ is contained in the single copy into the both factorials, namely: (a) in the numerator the second copy $3q > 2n$; (b) in the denominator we have $2q > 2 \times \frac{2}{3}n = n + \frac{1}{3}n > n+1$ (let's remember $n \geq 4$).

As a result, the first (probably not the last) no-go zone is defined in the upper base, i.e. $F_n \cap_p [2/3 n, n+1] = \emptyset$. Additionally, we have lowered the border $\text{ppb}(n) < 2/3 n$.

Let's do a final check of any prime $1/2(n+1) < p < 2/3 n$. In the numerator we get the second copy of thanks to an odd composite number $3p < 3 \times 2/3 n = 2n$. In the denominator p remains in a single copy ($2p > n+1$), so $p \in F_n$. Clearly, these primes form the next Chebyshev's segment. Let foregoing be regarded as a proof of the following theorem.

Theorem 2.1. *Let prime $p \in_p [2/3 n, n+1] \subset \mathbb{P}$. Then $p \nmid C(n)$.*

Two segments and the no-go zone between them is reduced fourfold the space of the factor base for a selection of the remaining primes.

Power border. Moving down the factor base, starting from the main segment and passing through the no-go zone, we found the next Chebyshev's segment. Obviously, there is another no-go zone lower. So we will be a long time to reach the power border. The following example will help accelerate this process.

Example 2.1. Let's choose the prime 101 and let's try to find the smallest index n such that $101^2 \mid C(n)$. In this case power border is 101 or very close. This experiment will help to further define the power border for an arbitrary Catalan number. In (4) let's write some composite factors of the numerator and of the denominator which are multiple of 101:

numerator 1, 3, 5, 7, ..., 101, ..., 101×3, ..., 101×5, ..., 101×101, ...
denominator 1, 2, 3, 4, ..., 101, ..., 101×2, ..., 101×3, ..., 101×101, ...

In the case of $n \leq 50$ the both factorials don't contain 101. So $101 \nmid C(n)$, $n \leq 50$. In the following range $51 \leq n < 100$ the factor 101 is only in the numerator in a single copy.

Thus, we found the zone of favorable indexes for 101, i.e. $101 \mid C(n)$, $51 \leq n < 100$. For $n=100$ the symmetric factor 101 appears in the denominator, therefore, $101 \nmid C(100)$. This situation will persist in the range $100 \leq n < 152$. This is the no-go zone indexes for 101.

Gradually increasing an index n , we get a series of favorable and no-go zones. In each favorable zone $101^2 \nmid C(n)$, and when we get to $2n-1 = 101 \times 101$, then we obtain two factors 101 in the numerator at once. Thus, in the case of $n = (101^2 + 1)/2 = 5101$ the prime power 101^2 divides $C(5101)$. So, we got a required Catalan number.

In the Example 2.1 we received the smallest 5101th Catalan number that is divisible by 101^2 . For the next prime 103 we get another index $n = (103^2 + 1)/2 = 5305$. So, $p^2 \nmid C(5101)$, $p > 101$, and therefore you can take as a power border, for example, the next even integer, i.e. $\text{ppb}(5101) = 102$. We have the right to formulate the following theorem.

Theorem 2.2. *Let prime $p > \sqrt{2n}$, then $p^k \nmid C(n)$, $k \geq 2$.*

It is logical and convenient the power border to take equal to $\text{ppb}(n) = \sqrt{2n}$. An integer $\text{ppb}(n)$ is always an even number (in this case, n is also an even number), so $\text{ppb}(n) \notin \text{FB}(n)$. Let's give a more general theorem.

Theorem 2.3. *Let prime $p > \sqrt[m]{2n}$, then $p^k \nmid C(n)$, $k \geq m$.*

Remark 2.1. When we say, for example, about an even number $\frac{2}{3}n$ or a natural number $(n+1)/2$, we mean certain (or probable) values for n , namely: in the first case, n is divisible by 3, the second n is an odd number.

3. Train of Catalan numbers

For the n th Catalan number the border $\text{ppb}(n) = \sqrt{2n}$ divides the base $\text{FB}(n)$ into two disjoint subsets. The upper base $\text{UB}(n) = {}_p(\text{ppb}(n), 2n)$ is a group of Chebyshev's segments S_i , that are separated by no-go zones Z_j . Chebyshev's segments accumulate the most primes that form a train of the Catalan number. For example, the train of the 1,000,000th Catalan number contains 99.7% of all prime factors.

The main segment is the longest, adjacent to the top of $\text{FB}(n)$, is evident in analytical formulas and therefore is defined in the first turn. So assign the main segment of the first number, i.e. $S_1 = {}_p(n+1, 2n)$. The bottom S_1 is adjacent the most extensive no-go zone, this zone will also give the first number, i.e. $Z_1 = {}_p[\frac{2}{3}n, n+1]$. In addition, we discovered another Chebyshev's segment $S_2 = {}_p(\frac{1}{2}(n+1), \frac{2}{3}n)$.

In this section we will consider the layout of segments and zones in the upper base $\text{UB}(n)$. Try to estimate the total number of segments and zones.

Top of Chebyshev's segments. Previously we found an upper open bound of the second segment $\frac{2}{3}n$. Probable Prime number $\frac{2}{3}n-1 \in S_2$ (in this case $3|n$). If we write the top open bound of segment S_1 in the form $2n/1$, we can trace certain regularity. Apparently, the upper open bound of S_3 is $2n/5$, the next segment will receive the value $2n/7$, etc. Check it on the example.

Example 3.1. Let $n=7500$, then the top open bound of the 3rd segment must be equal to $2 \times 7500 / (2 \times 3 - 1) = 3000$. Consider the neighboring primes 2999 and 3001 on both sides of the bound. Refer again to the formula (4).

In the odd double factorial $14999!!$ (the numerator of the fraction) the prime 2999 is in triplicate, namely: $2999, 3 \times 2999, 5 \times 2999$. But in the ordinary factorial $7501!$ (the denominator of the fraction) there are only 2999 and 2×2999 . So $2999 \in F_{7500}$. But for the next prime 3001 the composite number of the numerator 5×3001 is outside of $\text{FB}(7500)$, so in the numerator and in the denominator there are exactly two copies of 3001. Therefore, $3001 \notin F_{7500}$.

The technique that used in Example 3.1 can be applied to check the top open bound of arbitrary Chebyshev's segment. Generalize the obtained result in the following theorem (the proof is in the next section).

Theorem 3.1. *In the train of the n th Catalan number the top open bound of k -th Chebyshev's segment is $2n/(2k-1)$, $k \geq 1$.*

Bottom of Chebyshev's segments. Let us write the lower open bound of the main segment in the form to $(n+1)/1$. Following the second Chebyshev's segment has a bottom bound $(n+1)/2$. Probably for the third segment we need to get the value $(n+1)/3$, etc. The trend is obvious and we can formulate a corresponding theorem (the proof, see the next section).

Theorem 3.2. *In the train of the n th Catalan number the bottom open bound of k th Chebyshev's segment is $(n+1)/k$, $k \geq 1$.*

So for the n th Catalan number, the train components take the following form:

$$S_k = {}_p((n+1)/k, 2n/(2k-1)), \quad Z_k = {}_p[2n/(2k+1), (n+1)/k], \quad k \geq 1. \quad (5)$$

How many segments in the train of the n th Catalan number? To estimate this value easily. The lower bounds of the k th Chebyshev's segments should be above the power border, i.e. $(n+1)/k > \text{ppb}(n)$ or

$$k < (n+1)/\text{ppb}(n) \approx n/\text{ppb}(n) = \frac{1}{2}\text{ppb}(n) = \sqrt{n/2}.$$

Let us formulate the corresponding theorem.

Theorem 3.3. *The number of the Chebyshev's segments in the train of the n th Catalan number is less than $\sqrt{n/2}$.*

The actual number of Chebyshev's segments is often substantially less. The bounds of the segments converge rapidly as it approaches the power border, and many segments (and no-go zones also) near the power border are "collapse", i.e. is empty, without primes.

The bounds of segments are fractional numbers in most cases. It is more convenient to work with integer values, but rounding intervals should be kept open bounds. In the bottom of the segment will truncate the fractional part (rounding down "the floor"), and for the top is taken as the nearest integer from above (rounding up "the ceiling").

Example 3.2. Consider the prime factorization of the millionth Catalan number (in the last section there are real programs for test calculations). The natural form of this number is not interesting, because the array of 600 thousand decimal digits has little to say. It's different with primes, even if a lot of them. Multiset $F_{1000000}$ includes 101543 primes and prime powers. All primes are selected from the factor base ${}_p(1, 2 \cdot 10^6)$.

The base separated by the border $\text{ppb}(10^6) = \sqrt{2 \times 10^6} = 1414.21$ into two disjoint areas. In the upper base there are 707 Chebyshev's segments, which contain 99.7% of all prime factors. For example, in the segment $S_1 = {}_p(1000001, 2 \cdot 10^6)$ there are 70435 primes, and the second $S_2 = {}_p(5 \cdot 10^5, 666667)$ includes 12531 primes.

The nearest prime 1423 above the power border falls into the segment $S_{703} = {}_p(1422, 1424)$. Consequently, four segments 704-707 are empty, i.e. don't contain primes. The next prime $1433 | C(10^6)$ select by the segment $S_{698} = {}_p(1432, 1434)$. As a result, we have

another group of empty segments numbered 699-702. As you can see, near the power border only the fifth segment "brings prey", i.e. it contains a prime that divides $C(10^6)$.

Example 3.3. For the 100,000,000th Catalan number $\text{ppb}(10^8) = 14142$, so the primes of the train are selected from the open interval ${}_p(14142, 2 \cdot 10^8)$. Let's check two primes $p = 463,219$ and $q = 543,061$. To solve this problem it is enough to compare these numbers with the bounds of several Chebyshev's segments.

Choose a suitable segment for the first prime. We use the formula lower bounds: $(10^8 + 1)/463219 = 215.88$. The prime p is below the segment 215, but above the lower limit of the segment 216. It remains to compare p with the upper limit of the segment 216, and this value is equal to $\lceil 2 \cdot 10^8 / (2 \cdot 216 - 1) \rceil = 464,038 > p$. So, $p \in S_{216}$.

Note that, we can obtain an integer in the formula of the lower bound. This is rare, but possible. For example, for the 99,592,084th Catalan number we get the integer value $(99592084 + 1)/463219 = 215$, i.e. prime 463219 is the lower bound of 215th segment. In this case the prime is rejected immediately, since it is not included in the decomposition.

Similar calculations show that the second prime q is placed into the no-go zone between S_{184} and S_{185} . Consequently, $543061 \nmid C(100,000,000)$.

Checking, as described in Example 3.3, is reduced to the following two successive steps: (a) calculate the segment number, the lower limit of which is closest to the bottom of a prime p ; (b) compare p with the upper limit of the found segment. Let's state an important theorem, which makes it easy to check any prime number.

Theorem 3.4. *The prime $p > \sqrt{2n}$ divides the n th Catalan number if and only if $p < \lceil 2n / (2k - 1) \rceil$, where $k = \lceil (n + 1) / p \rceil$, $kp \neq n + 1$.*

The following corollary follows immediately from Theorem 3.4.

Corollary 3.1. *The n th Catalan number is not divisible by prime $p > \sqrt{2n}$ if p divides $n + 1$.*

Sieve of Chebyshev. To get the Catalan number train we can move the contents of the Chebyshev's segments in empty set, for example starting from the main segment. But there is a faster method, which is similar to the algorithm for constructing the famous Sieve of Eratosthenes, when composite numbers which are divisible by primes 2, 3, 5, etc., are deleted sequentially from the set of natural numbers. Obviously, it is easier and faster to clear all no-go zones at the upper factor base, keeping the contents of the segments. The result is a construct that by analogy we will call the *Sieve of Chebyshev*.

4. The proofs of Theorem 3.1 and Theorem 3.2

Let us introduce some definitions and notation that will be needed in the future.

Power function. In the literature we can find the *power function* $v_p(m)$ that denotes the largest power of a prime p dividing a positive integer m [Pom13]. For example, $v_7(14) = 1$, $v_7(98) = 2$, $v_7(5!) = 0$. Here are a few properties of the power function:

$$v_p(ab) = v_p(a) + v_p(b), \quad v_p(a^k) = kv_p(a), \quad v_p(a/b) = v_p(a) - v_p(b).$$

Various properties of the power function see [Ep15]. The power function is used in a Prime Factorization of special numbers. For example, the power of the odd prime p in the Cayley formula (2) is given by:

$$v_p(C(n)) = v_p((2n-1)!!) - v_p((n+1)!), \quad p > 2. \quad (6)$$

The last operand in (6) can be calculated using Legendre's formula [Pom13]:

$$v_p((n+1)!) = \sum_{j>0} \lfloor (n+1)/p^j \rfloor. \quad (7)$$

Odd rounding "floor". For a positive $x > 1$, let $\lfloor x \rfloor_{\text{odd}}$ denote the rounding "floor" to the nearest odd positive integer, i.e. the fractional part of x is discarded with an additional decrease to 1 if the result is an even number. For example, $\lfloor 23/7 \rfloor_{\text{odd}} = \lfloor 34/7 \rfloor_{\text{odd}} = 3$. Let's call this operation an *odd rounding "floor"*.

This operation is useful to determine the power of a prime in the double factorial. For example, the first operand in (6) can be calculated as follows (see [Ep15]):

$$v_p((2n-1)!!) = \frac{1}{2} \sum_{j>0} (\lfloor (2n-1)/p^j \rfloor_{\text{odd}} + 1). \quad (8)$$

The proof of Theorem 3.1. It is necessary to show that the k th segment of the n th Catalan number has the open top bound $u = 2n/(2k-1)$. When $k=1$ we get the top base $2n$, since the main segment is adjacent to the open top of $\text{FB}(n)$.

Let u be an integer and then u is an even number. If is not, and u is a fractional number, then round u to the nearest even number. Let $p = u-1$ and $q = u+1$ be primes. Obviously, $\text{ppb}(n) = (2n)^{\frac{1}{2}} < p < u < q$. So, we chose the most unfavorable conditions for p, u, q . Let us prove that $p \mid C(n)$ and $q \nmid C(n)$. Obviously,

$$n = \frac{1}{2}u(2k-1) = \frac{1}{2}(p+1)(2k-1) = \frac{1}{2}(q-1)(2k-1).$$

Since $p^k > 2n$, $k > 1$, then the equalities (7-8) in each sum there is one term, i.e.

$$v_p((n+1)!) = \lfloor (n+1)/p \rfloor \quad \text{and} \quad v_p((2n-1)!!) = \frac{1}{2} (\lfloor (2n-1)/p \rfloor_{\text{odd}} + 1), \quad p > (2n)^{\frac{1}{2}}.$$

The same is true for q . To prove the theorem it is enough to confirm the equalities

$$v_p(C(n)) = 1, \quad n = \frac{1}{2}(p+1)(2k-1), \quad \text{and} \quad v_q(C(n)) = 0, \quad n = \frac{1}{2}(q-1)(2k-1). \quad (9)$$

Let us first consider the ordinary factorials from (6).

$$\begin{aligned} v_p((n+1)!) &= \lfloor (\frac{1}{2}(p+1)(2k-1) + 1)/p \rfloor \\ &= \lfloor k + k/p - \frac{1}{2} + \frac{1}{2p} \rfloor = \begin{cases} k-1, & \text{if } k < p/2. \\ k, & \text{if } k \geq p/2. \end{cases} \end{aligned} \quad (10)$$

$$\begin{aligned} v_q((n+1)!) &= \lfloor (\frac{1}{2}(q-1)(2k-1) + 1)/q \rfloor \\ &= \lfloor k - k/q - \frac{1}{2} + \frac{3}{2q} \rfloor = \begin{cases} k-1, & \text{if } k \leq q/2. \\ k-2, & \text{if } k > q/2. \end{cases} \end{aligned} \quad (11)$$

The operands $^{1/2}p$ and $^{3/2}q$ removed, since they do not affect the final result. In (10-11) only the top branch are real, so $v_p((n+1)!) = v_q((n+1)!) = k-1$.

Let's calculate the odd double factorials from (6).

$$\begin{aligned} v_p((2n-1)!!) &= \frac{1}{2}(\lfloor ((p+1)(2k-1)-1)/p \rfloor_{\text{odd}} + 1) = \\ &= \frac{1}{2}(\lfloor 2k-1 + ^{2/p}(k-1) \rfloor_{\text{odd}} + 1) = \frac{1}{2}((2k-1) + 1) = k. \end{aligned}$$

$$\begin{aligned} v_q((2n-1)!!) &= \frac{1}{2}(\lfloor ((q-1)(2k-1)-1)/q \rfloor_{\text{odd}} + 1) = \\ &= \frac{1}{2}(\lfloor 2k-1 - 2k/q \rfloor_{\text{odd}} + 1) = \frac{1}{2}((2k-3) + 1) = k-1. \end{aligned}$$

Recent calculations confirm (9). The proof of the Theorem 3.1 is complete.

The proof of Theorem 3.2. The open bottom of the Chebyshev's segment unlike the open top bound may be odd prime. In this regard, we show first that

$$p = (n+1)/k > \text{ppb}(n), \quad k < ^{1/2}\text{ppb}(n),$$

does not divide $C(n)$. In other words, by means of (6) let us prove the equality

$$v_p((2n-1)!!) = v_p((n+1)!), \quad n = kp-1. \quad (12)$$

Factorials (7-8) are free from squares p , therefore

$$\begin{aligned} v_p((n+1)!) &= \lfloor (kp)/p \rfloor = k, \\ v_p((2n-1)!!) &= \frac{1}{2}(\lfloor (2kp-3)/p \rfloor_{\text{odd}} + 1) = \frac{1}{2}(\lfloor 2k-3/p \rfloor_{\text{odd}} + 1) = \frac{1}{2}(2k-1+1) = k. \end{aligned}$$

So, $(n+1)/k \nmid C(n)$. Next, check the probably least prime of the k th segment. Let $w = (n+1)/k$ be an even integer, and let $q = 1+w < 2n/(2k-1)$ be a prime. We will show that $q \mid C(n)$. Obviously, $n = k(q-1)-1$, $q > w > \text{ppb}(n) > 2k$ or $q \geq 2k+3$. Let's compute factorials.

$$\begin{aligned} v_p((n+1)!) &= \lfloor k(q-1)/q \rfloor = \lfloor k - k/q \rfloor = k-1. \\ v_p((2n-1)!!) &= \frac{1}{2}(\lfloor (2(k(q-1)-1)-1)/q \rfloor_{\text{odd}} + 1) = \\ &= \frac{1}{2}(\lfloor 2k - (2k+3)/q \rfloor_{\text{odd}} + 1) = \frac{1}{2}(2k-1+1) = k. \end{aligned}$$

Thus, in the k th Chebyshev's segment the probably least prime $1+(n+1)/k$ is guaranteed divides $C(n)$. The proof of the Theorem 3.2 is complete.

5. Slight Catalan numbers

The prime factors of a Catalan number that exceed the power border choose from the set of primes based on the segment bounds. Thus, the vast majority of primes are known, and Chebyshev segments can't be stored. The train is ballast, it can be reset and if necessary easily restore. It is enough to keep the core of Catalan numbers. Below is selected information for the first Catalan numbers.

| Index | Power border | Light Catalan number | Prime factorization of Catalan number | | Catalan number «heavyweight» |
|-------|--------------|----------------------|---------------------------------------|---------------------------|---------------------------------|
| | | | core | train | |
| 3 | 2,45 | 1 | | | 5 |
| 4 | 2,83 | 2 | 2 | | 7 |
| 5 | 3,16 | 6 | 2, 3 | | 7 |
| 6 | 3,46 | 12 | 2, 2, 3 | | 11 |
| 7 | 3,74 | 3 | 3 | 11, 13 | |
| 8 | 4,00 | 2 | 2 | 5, 11, 13 | |
| 9 | 4,24 | 2 | 2 | 11, 13, 17 | |
| 10 | 4,47 | 4 | 2, 2 | 13, 17, 19 | |
| 11 | 4,69 | 2 | 2 | 7, 13, 17, 19 | |
| 12 | 4,90 | 4 | 2, 2 | 7, 17, 19, 23 | |
| 13 | 5,10 | 100 | 2, 2, 5, 5 | 17, 19, 23 | |
| 14 | 5,29 | 360 | 2, 2, 2, 3, 3, 5 | 17, 19, 23 | |
| 15 | 5,48 | 45 | 3, 3, 5 | 17,19, 23, 29 | |
| 16 | 5,66 | 90 | 2, 3, 3, 5 | 19, 23, 29, 31 | |
| 17 | 5,83 | 30 | 2, 3, 5 | 11, 19, 23, 29, 31 | |
| 18 | 6,00 | 300 | 2, 2, 3, 5, 5 | 7, 11, 23, 29, 31 | |
| 19 | 6,16 | 30 | 2, 3, 5 | 7, 11, 23, 29, 31, 37 | |
| 20 | 6,32 | 60 | 2, 2, 3, 5 | 7, 11, 13, 23, 29, 31, 37 | |

A light Catalan number is significantly shorter than its ancestor "heavyweight" for two reasons: firstly, the core is tiny in size and, secondly, the core contains the smallest factors. The difference is especially noticeable on large indices, for example, the 170th light Catalan number is 4080, while in the corresponding progenitor 99 digits:

$$C(170) = 566\ 348\ 408\ 726\ 522\ 751\ 148\ 449\ 775\ 858\ 720\ 556\ 691\ 622\ 190\ 005\ 859\ 389\ 137\ 334\ 807\ 741\ 187\ 245\ 136\ 563\ 759\ 042\ 704\ 144\ 561\ 379\ 440.$$

Of course, and light numbers on the big indices look impressive. The light 50,000th Catalan number has a length of 163 marks, here is this number:

1 029 142 440 334 210 758 480 708 051 574 203 810 960 548 889 204 183 685 792 756
 307 454 455 534 384 071 475 346 148 063 228 602 598 951 202 112 317 923 378 920 856
 767 137 362 573 866 245 850 058 506 779 182 608 960.

But "heavyweight" with the index 50,000 show is problematic because it "weighs" 30,000 digits. Among the light Catalan numbers there are "doubles".

The small sizes of the light numbers give the opportunity to learn the "depth" of the sequence of Catalan number. And more interesting core than the naturalized light Catalan number. Let's show the 100 light Catalan numbers (the first four numbers equal to 1):

00-09: 1, 1, 1, 1, 2, 6, 12, 3, 2, 2;

10-19: 4, 2, 4, 100, 360, 45, 90, 30, 300, 30;

20-29: 60, 60, 120, 450, 36, 1764, 392, 28, 280, 56;

30-39: 112, 7, 294, 1470, 84, 14, 28, 28, 1400, 490;

40-49: 980, 3780, 7560, 18900, 2520, 2520, 35280, 4410, 900, 36;

50-59: 216, 108, 216, 840, 336, 12, 24, 24, 240, 72;

60-69: 1008, 121968, 11616, 45375, 18150, 1650, 3300, 11550, 1039500, 29700;

70-79: 59400, 4950, 108900, 544500, 2134440, 1067220, 27720, 83160, 831600, 20790;

80-89: 1540, 10780, 21560, 84700, 33880, 5725720, 34354320, 780780, 273273000, 18218200;

90-99: 400400, 200200, 400400, 2002000, 8808800, 34684650, 69369300, 1415700, 5577000, 111540.

6. Software service

In conclusion, let us consider the software system for carrying out test calculations online (for more details see [Ep15]). Let's list some programs with brief description of their functions.

A small program can produce [a list of primes](#) in a certain range; the start of the range does not exceed 10^{10} . The prime factorization of the Catalan numbers can be performed in several ways. In [the first variant](#), the Catalan number is displayed in naturalized form (maximum index of 10^4). In [the second case](#), a natural number is not displayed (the index up to 10^7). There is [a third option](#), in which factorizes only the light Catalan number (the index up to 10^8). A separate program allows you to get [the core](#) of the Catalan number (the index is practically unlimited).

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